

# Renormalization for algebraic geometers

(1)

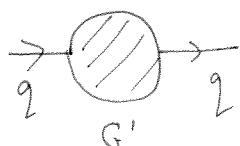
I). Subdivergence-free graphs

II). Renormalization of 1-scale graphs.

III). Graph & Numbers. Witten map      Graphs  $\longrightarrow$  Numbers.

Requires single-scale processes. There are several possibilities

Ex: Scalar massless QFT ( $\text{eg } \phi^4$ )

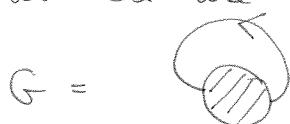


$$q \in \mathbb{R}^4$$

$G'$  subdivergence free  $\leadsto$  contribution to 2-pt func

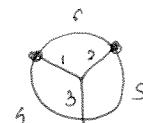
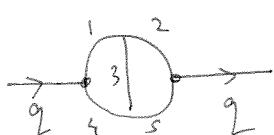
Trivial depends on  $q^2$ ; depends a number only.

This number can also be obtained from 4-pt function by closing external legs.



$$\sim \text{number} \times \log\left(\frac{q^2}{\mu^2}\right)$$

Ex: Massless 2-loop 2pt function (Brodsky, Weizsäcker)



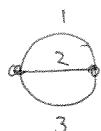
$$6J(3)$$

Defn: Graph polynomial  $\psi_G \in \mathbb{Z}[\alpha_e, e \in E(G)]$  ( $G$  connected graph, no external legs)

$$\psi_G = \sum_{\substack{T \subseteq G \\ \text{s. tree}}} \prod_{e \notin T} \alpha_e$$

$T$  spanning tree =  $T \subseteq G$  st  
 $T$  connected,  $h_1(T) = 0$  &  $V(T) = V(G)$

Ex:



$$\psi_G = \alpha_1 \alpha_2 + \alpha_1 \alpha_3 + \alpha_2 \alpha_3$$

$$\boxed{\deg \psi_G = h_G}$$

$G \in \phi^*$  if degree of each vertex  $\leq 4$ .

Superficial degree of divergence

$$sd(G) = 2h_G - |E_G|$$

$G$  is 

- convergent if  $sd(G) < 0$

- log-divergent if  $sd(G) = 0$

- quadratically divergent if  $sd(G) = 1$

$G$  is overall log-div., sub-divergence free if  $\begin{cases} \cdot sd(Y) = 0 \\ \cdot sd(Y) < 0 \quad \forall Y \neq G \end{cases}$

Amplitudes Let  $\mathcal{L}_G = \sum_{i=1}^{E_G} (-1)^i \alpha_i \overset{\wedge}{d\alpha_1 \wedge \dots \wedge d\alpha_{i-1} \wedge d\alpha_{i+1} \wedge \dots \wedge d\alpha_{E_G}}$

$$I_G = \int_{\Delta} \frac{\mathcal{L}_G}{\omega_G^2} \in \mathbb{R} \quad \text{converges (Weingärtner) when } G \text{ overall log-div, sub-div free.}$$

$$\Delta = \{(\alpha_1, \dots, \alpha_{E_G}) \in \mathbb{P}^{E_G-1}(\mathbb{R}) : \alpha_i \geq 0\} \quad \text{simplex.}$$

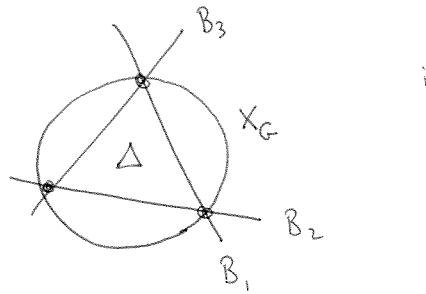
## 2) Graph hypersurfaces & blow-ups (Bloch-Esnault-Kreimer)

Graph hypersurface :  $X_G \subseteq \mathbb{P}^{E_G-1}$  highly singular.

$$X_G = V(\omega_G)$$

$$\partial \Delta \subseteq B \quad B = V(\alpha_1, \alpha_2, \dots, \alpha_{E_G}) \subseteq \mathbb{P}^{E_G-1} \quad B = \bigcup B_i$$

E.g.:  $G = \bigoplus_3$



ideal simplex in hyperbolic 2-space  
⇒ ∞ volume!

~~X\_G~~  $B \cup X_G$  not single normal crossing

(i) Bad loci. If ~~bad loci~~  $I \subseteq E(G)$ , let  $B_I = \bigcap_{i \in I} B_i$ .

Let  $D_I = \{ \alpha_i = 0 \mid i \in I, \alpha_i > 0 \mid i \notin I \} \subseteq \mathbb{P}^{E_G-1}(\mathbb{R})$

$$\begin{array}{ccc} I \subseteq E(G) & \longleftrightarrow & \text{subgraphs } I \subseteq G \\ B_I \text{ or } D_I & & \end{array}$$

Claim :  $[X_G \cap D_I \neq \emptyset] \iff [D_I \subseteq X_G] \iff [h_I > 0]$

$G = \bigoplus_3$

$$\omega_G = \alpha_1 \alpha_2 + \alpha_1 \alpha_3 + \alpha_2 \alpha_3 \geq 0 \text{ on } D.$$

PF:  $\psi_0 = 0$  on  $D_I \iff$  all merenials  $T_{\text{eff}}^I$  vanish on  $D_I$   
 $\iff$   $\nexists$  spanning trees  $T$ ,  $T^c \cap I \neq \emptyset$   
 $\iff$   $I$  not contained in any spanning tree  
 $\iff h_I > 0$ .

"Bad loci are contained in the linear subspaces only"

(ii) Blow-ups: de Concini-Procesi / Fulton-Margheron recipe for blowing up linear subspaces: blow up in increasing order of dimension. Result is independent of the chosen order.

$$\text{Bad loci} = \{ L_I \subseteq \mathbb{P}^{E_G-1} \text{ where } I \text{ minimal, } h_I > 0 \}$$

Generates a poset by taking intersections

$$L_{I_1} \cap \dots \cap L_{I_k} = L_{I_1 \cup \dots \cup I_k}$$

Blow up: points, lines, planes, ...

$$\pi_G : P_G \longrightarrow \mathbb{P}^{E_G-1}$$

Let  $Y_G =$  strict transform of  $X_G$  graph hypersurface

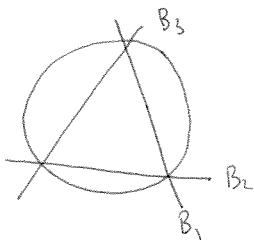
$\tilde{B}' =$  total transform of  $B = \bigcup B_i \cup$  Exceptional divs.

Exceptional divs:  $E_I \longleftrightarrow I \subseteq G$  minimal subgraph  $h_I > 0$ .

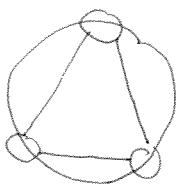
Let  $\tilde{\Delta}_G = \overline{\pi_G^{-1}(B)}$  (usual topology) Feynman polytope.

Prop (BEK)  $Y_G \cap \tilde{\Delta}_G = \emptyset$ .

Ex: ①



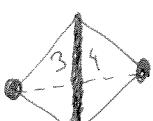
3 min. graphs with loops



$\tilde{\Delta}_G =$  hexagon.

②

3 min. graphs



$\rightarrow$



$\tilde{\Delta}_G$

$$\omega_G = \frac{\mathcal{I}_G}{\psi_G^2} \quad \mathcal{I}_G = \int_{\Delta_G} \omega_G \quad \text{converges} \iff \int_{\tilde{\Delta}_G} \pi^*(\omega_G) \quad \text{converges}.$$

$\tilde{\Delta}_G$  compact,  $\Leftrightarrow \pi^*(\omega_G)$  has no poles along  $\mathcal{E}_{\mathcal{I}}$ .

### 3). Residues & Operads

#### (i) Product structure

Key identity:  $G$  graph,  $\gamma \subseteq G$  subgraph,  $G/\gamma$  contract all components of  $\gamma$ .

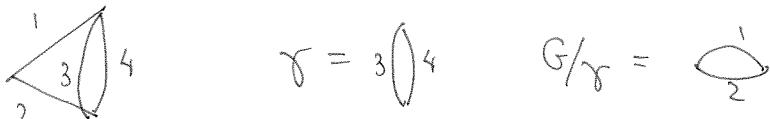
$$\Rightarrow \boxed{\psi_G = \psi_{\gamma} \psi_{G/\gamma} + R_{\gamma}}$$

where  $R_{\gamma} \in \mathbb{Q}[\alpha_e, e \in \gamma] \otimes_{\mathbb{Q}} \mathbb{Q}[\alpha_e, e \in G/\gamma]$   $\deg_{\gamma} R_{\gamma} > h_{\gamma}$ .

i.e.  $\alpha_e \mapsto t \alpha_e \quad \forall e \in \gamma$

$$\psi_G \sim t^{h_{\gamma}} \cdot \psi_{\gamma} \psi_{G/\gamma} + O(t^{h_{\gamma}+1})$$

Ex:



$$\psi_G = \underbrace{(\alpha_3 + \alpha_4)}_{\psi_{\gamma}} \underbrace{(\alpha_1 + \alpha_2)}_{\psi_{G/\gamma}} + \underbrace{\alpha_3 \alpha_4}_{R_{\gamma}} \sim (\alpha_1 + \alpha_2)(\alpha_3 + \alpha_4)$$

Prop:  $\pi^*(\omega_G)$  has pole along  $\mathcal{E}_{\mathcal{I}}$  of order  $\text{sd}(\mathcal{I}) + 1$ .

$\mathcal{I}$  conv.  $\Leftrightarrow$  no pole

log div  $\Leftrightarrow$  simple pole

quad.  $\Leftrightarrow$  double pole.

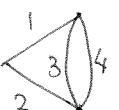
Cor:  $G$  subdivision free, overall log. div  $\Rightarrow \mathcal{I}_G < \infty$ .

Geometrically,  $\mathcal{E}_I \cong \mathbb{P}^{I|I-1} \times \mathbb{P}^{I|G|I-1}$   
Lemma (BEV) :  $Y_G \cap \mathcal{E}_I \cong (X_I \times \mathbb{P}^{I|G|I-1}) \cup (\mathbb{P}^{I|I-1} \times X_{G/I})$ .

Now suppose  $G$  has at most log-divergent subgraphs. Then we have residue map along  $\mathcal{E}_I$

$$\text{Res} \Big|_{\mathcal{E}_I} : \Omega^*(P_G, Y_G) \longrightarrow \Omega^*(P_I, Y_I) \otimes \Omega^*(P_{G/I})$$

Claim :  $\text{Res} \Big|_{\mathcal{E}_I} \pi^*(\omega_G) = \begin{cases} \pi^*(\omega_I) \otimes \pi^*(\omega_{G/I}) & \text{sd}(I) = 0 \\ 0 & \text{else.} \end{cases}$

Ex:   $\pi^*(\omega_G)$  single pole along  $\mathcal{E}_I$   $\gamma = 30^\circ$ .

Compute blow-up : on affine  $x_1 = 1$

$$\omega_G = \frac{dx_2 dx_3 dx_4}{[(1+\alpha_2)(\alpha_3 + \alpha_4) + \alpha_3 \alpha_4]^2} \quad \beta_3 = \alpha_3, \quad \beta_4 = \frac{\alpha_4}{\alpha_3}, \quad \beta_2 = \alpha_2$$

$$\pi^*(\omega_G) = \frac{\beta_3 d\beta_2 d\beta_3 d\beta_4}{\beta_3^2 [(1+\beta_2)(1+\beta_4) + \beta_3 \beta_4]^2} \quad \text{single pole at } \beta_3 = 0$$

$$\text{Res} \Big|_{\beta_3=0} \pi^*(\omega_G) = \frac{d\beta_2}{(1+\beta_2)} \otimes \frac{d\beta_4}{(1+\beta_4)}$$

□

## (ii) Operads.

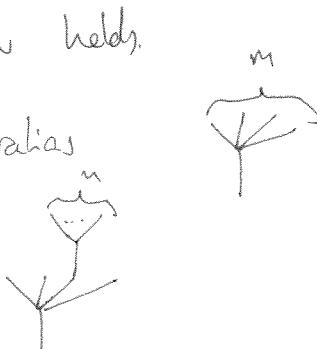
Naive defn : An operad is a collection of (vector spaces,  $\{P(n)\}_{n \in \mathbb{N}}$ ) in a monoidal category  $\text{Sch}/\mathbb{Z}$  st

$$o_i : P(m) \times P(n) \longrightarrow P(m+n-1) \quad \text{is composition law.}$$

st : ~~associates~~ assoc. law holds.

View elts of  $P(m)$  as  $m$ -ary operations

$o_i$  = graft into  $i^{\text{th}}$  branch



Dual operad: cooperad  
 $\Delta_i : P(m+n-1) \longrightarrow P(m) \otimes P(n)$ .

Any operad defines a pre-Lie structure

$$A \circ B = \sum_i A \circ_i B$$

antisymmetrizing gives Lie algebra structure  $\{A, B\} = A \circ B - B \circ A$   
 on  $\bigoplus_{n \in \mathbb{N}} P(n)$ .

(iii) Graphs:  $\{(P_G, Y_G)\}_G$  "operad in  $\text{Sch}/\mathbb{Z}$ ".

$$\text{Maps: } P_\gamma \times P_{\gamma/\gamma} \cong E_\gamma \subseteq P_G$$

$$(P_\gamma, Y_\gamma) \times (P_{\gamma/\gamma}, Y_{\gamma/\gamma}) \longrightarrow (P_G, Y_G) \quad \text{morphism.}$$

$$\text{Dually: } \mathcal{R}^{\circ}_{\log}(P_G, Y_G) \xrightarrow{\text{Res}} \mathcal{R}^{\circ}_{\log}(P_\gamma, Y_\gamma) \otimes \mathcal{R}^{\circ}_{\log}(P_{\gamma/\gamma}, Y_{\gamma/\gamma})$$

"cooperad".

$$\begin{array}{ccc} \{ G \text{ log div with } \\ \text{log div sub div} \} & \dashrightarrow & \{ \text{Pairs } (P_G, Y_G), \\ & & \text{including int boundary} \} & \xrightarrow{\omega_0} & \{ \text{ } \mathcal{R}^{\circ}(P_G, Y_G) \\ & & & & \text{Feynman rules} \\ & & & \text{operad} & \text{cooperad} \end{array}$$

Take Lie alg / Lie coalg

$\implies$  Gours-Kremer coproduct / core product

$$\Delta_{\text{core}} : G \longmapsto \sum_{\substack{\gamma \in G \\ \gamma \text{ int} \\ h_\gamma > 0}} \gamma \otimes G/\gamma$$

But w. factors through

$$\Delta_{\text{dk}} : G \longmapsto \sum_{\substack{\gamma \in G \\ \text{1PI, sd}(\gamma)=0}} \gamma \otimes G/\gamma$$

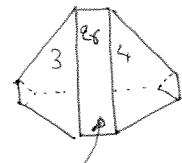
The Res map  
is more fundamental!

Summary :  $H = \mathbb{Z}[\text{Graph } G \text{ log div, with log-div s.d.}]$ .

$$H \xrightarrow{\Delta} H \otimes H$$

$$G \in H \xrightarrow{\text{down-up}} (\rho_G, \gamma_G) \xrightarrow[\text{Feyn. rule}]{\omega_0} \Omega^*(\rho_G, \gamma_G) \xrightarrow[\text{ill-defined}]{\int_{\tilde{\delta}_G}} I_G$$

Ex :  $G = \begin{array}{c} 1 \\ \diagup \quad \diagdown \\ 3 \quad 4 \\ \diagdown \quad \diagup \\ 2 \end{array}$



single pole along  $\mathcal{E}_G$

$$\text{Residue} = \omega_{\mathcal{E}_G} \otimes \omega_G|_{\mathcal{E}_G}$$

$$\Delta G = 1 \otimes G + G \otimes 1 + \gamma \otimes G/\gamma$$

$I_G$  diverges: how to make it finite next time.

#### ④ Mixed Hodge structures, weights, coaction...

$$(\rho_G, \gamma_G) \longrightarrow H^{E_G-1} \left( \rho_G, \gamma_G, \frac{\tilde{B} \cdot (\tilde{B} \cap Y_G)}{H_G} \right)$$

$\tilde{B} = B \cup \text{Exceptional divisors.}$

"Graph motive"  
Bloch-Beilinson-Katz

$$\text{use } [\tilde{\Delta}_G] \in \text{gr}^w_0(H_G^\vee, \tilde{B})$$

Q1. Where does  $[\pi^*(\omega_G)]$  sit??

$$[\pi^*(\omega_G)] \in W_m H_{G, \text{dR}} \Rightarrow "I_G \text{ of weight } \leq m".$$

Q2 : MHS is a Tamari category  $\Rightarrow$  equivalent to  $\text{Rep } G^{\text{MHS}}$ .

$\Rightarrow H_G$  is a representation of  $G^{\text{MHS}}$  ~~as~~ affine group scheme.

How to compute the action? Is  $\{[H_G, \pi^*(\omega_G), \Delta_G]\}$  preserved by  $G^{\text{MHS}}$ ?

• By the integrals  $I_G$  ~~lift~~ to a Hecke algebra of motivic periods?

## Derivation of algebraic Feynman rules

The Schwinger trick provides the integral

$$\int_0^\infty \frac{e^{-\frac{\phi_F(q)}{4\Gamma}x}}{\psi_F^2} \pi dx$$

which diverges if  $\text{sd}(\Gamma) \geq 0$ .

Lemma:

$$\boxed{\int_c^\infty \frac{e^{-tx}}{t} dt = -\log(c) - \log(x) + \gamma + O(c)}$$

Subtract at renormalization point  $q^2 = \mu^2$ . Since  $\Gamma$  style-scale, we get

$$\int_0^\infty \frac{e^{-\frac{\phi_F(q^2)}{4\Gamma}x}}{\psi_F^2} - \frac{e^{-\frac{\phi_F(\mu^2)}{4\Gamma}x}}{\psi_F^2} \pi dx$$

Scale variables  $x \mapsto t \propto x$ . Since  $\deg\left(\frac{\phi_F}{4\Gamma}\right) = 1$ , we get

$$\int_0^\infty \left( \lim_{c \rightarrow \infty} \int_c^\infty \frac{e^{-\frac{\phi_F}{4\Gamma} q^2 \cdot t}}{\psi_F^2} \frac{dt}{t} - \frac{e^{-\frac{\phi_F}{4\Gamma} \mu^2 t}}{\psi_F^2} \frac{dt}{t} \right) \pi dx$$

$\underbrace{\hspace{10em}}$

$$\frac{\log\left(-\frac{\phi_F}{4\Gamma} q^2\right) - \log\left(-\frac{\phi_F}{4\Gamma} \mu^2\right)}{\psi_F^2}$$

Writing  $s = \frac{q^2}{\mu^2}$ , we get

$$\int_0^\infty \frac{\sqrt{s}}{\psi_F^2} \cdot \log(s) \pi dx \quad \text{as required}$$

Derivation of center terms. Differences come from  $\gamma \otimes T/\gamma$ .

In expansion setting, center term in BPHZ is

$$\frac{e^{-\frac{\phi_r}{4\pi} \mu^2}}{4\gamma^2} \cdot \frac{e^{-\frac{\phi_{T/r}}{4\pi} q^2}}{4T/\gamma}$$

BPHZ tells us to subtract at  $q^2 = \mu^2$ :

$$\frac{e^{-\frac{\phi_r}{4\pi} \mu^2}}{4\gamma^2} \frac{e^{-\frac{\phi_{T/r}}{4\pi} q^2}}{4T/\gamma} - \frac{e^{-\frac{\phi_r}{4\pi} \mu^2}}{4\gamma^2} \frac{e^{-\frac{\phi_{T/r}}{4\pi} \mu^2}}{4T/\gamma}$$

Do the same trick as before:

$$\log \left( \frac{\frac{\phi_r}{4\pi} \mu^2 + \frac{\phi_{T/r}}{4\pi} q^2}{\frac{\phi_r}{4\pi} \mu^2 + \frac{\phi_{T/r}}{4\pi} \mu^2} \right)$$

$$\rightarrow \int \frac{\log \left( \frac{s \phi_r 4\pi \gamma + \phi_r \phi_{T/r}}{\phi_r \phi_{T/r} + \phi_{T/r} \phi_r} \right)}{4\gamma^2 4^2 T/\gamma} \quad \text{?}$$

At  $s=1$ , the log vanishes and gives zero. Therefore no information is lost if we apply  $s \frac{\partial}{\partial s}$ :

$$\boxed{\int \frac{\phi_r \phi_{T/r}}{4\gamma^2 4^2 T/\gamma (s \phi_r 4\pi \gamma + \phi_r \phi_{T/r})} \quad ?}$$

algebraic  
center-term.

This is, by definition,

$$\int \omega_{\gamma \otimes T/\gamma}(s)$$

## II. Renormalization of 1-scale graphs. (joint with D.Kreimer)

0). Recap :

- 1-scale process  $\xrightarrow{q} \bigcirc \xrightarrow{q}$  represented by 
- $H = \mathbb{Z}[\text{Graphs}]$  Graphs are log-divergent with at most log-divergent subgraphs:

$$sd(\gamma) \leq 0 \quad \forall \gamma \subseteq G, \quad (sd(\gamma) = 2h_\gamma - E_\gamma).$$

Multiplication = disjoint union

Cproduct:

$$\Delta: H \longrightarrow H \otimes H$$

$$G \longmapsto \sum_{\substack{\gamma \subseteq G \\ \text{SDI} \\ sd(\gamma)=0}} \gamma \otimes G/\gamma$$

- $\psi_G \in \mathbb{Z}[\otimes_e]$  graph polynomial

$$\omega_G = \frac{\Omega_G}{\psi_G^2} \quad \text{homogeneous degree } 0.$$

$$I_G = \int_{\Delta_G} \omega_G$$

$$\Delta_G = \{ (\alpha_1 : \dots : \alpha_{E_G}) : \alpha_i \geq 0 \} \subseteq \mathbb{R}^{E_G-1}.$$

$I_G$  diverges in general.

- Blow-up  $\pi_G: P_G \longrightarrow \mathbb{P}^{E_G-1}$   
Exceptional divisor  $E_I \longleftrightarrow$  subgraph  $I$  st  $h_I > 0$   
 $I$  minimal.

- $\pi^*(\omega_G)$  has at most simple poles along  $E_I$

$$\text{Res}_{E_I} \pi^*(\omega_G) = \pi^*(\omega_\gamma) \otimes \pi^*(\omega_{G/\gamma})$$

iff  $I = \gamma$   
subdivergent.  
 $(sd(\gamma) = 0)$

$\Rightarrow I_G$  has u-v singularity near  $\alpha_e \rightarrow 0$   $e \in E(\gamma)$ .

Key Property :

$$\text{Res } \omega = (\omega \otimes \omega) \circ \Delta$$

where  $\text{Res} = \bigoplus_I \text{Res}_{E_I}$  the "total residue",  $\omega(\zeta) = 0$

Example :  $G = \begin{array}{c} 1 \\ \diagdown \quad \diagup \\ 3 \quad 4 \\ \diagup \quad \diagdown \\ 2 \end{array}$  unique div. subgraph  $\gamma = 3 \diagup 4$ .

$$\Delta G = 1 \otimes G + G \otimes 1 + \gamma \otimes G/\gamma$$

$$\omega_G = \frac{\mathcal{L}_G}{[(\alpha_1 + \alpha_2)(\alpha_3 + \alpha_4) + \alpha_3 \alpha_4]^2} \quad \text{has simple pole near } \alpha_3, \alpha_4 \rightarrow 0$$

$$\Rightarrow \text{Res}(\omega_G) = \omega_\gamma \otimes \omega_{G/\gamma} = \frac{\mathcal{L}_\gamma}{(\alpha_3 + \alpha_4)^2} \otimes \frac{\mathcal{L}_{G/\gamma}}{(\alpha_1 + \alpha_2)^2}$$

Today : use BPMZ to subtract off the residues by inclusion-exclusion.

## 2). Second Symmetric polynomial & 1-scale graphs.

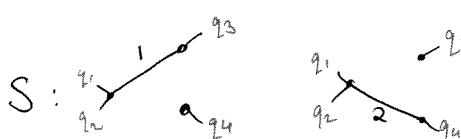
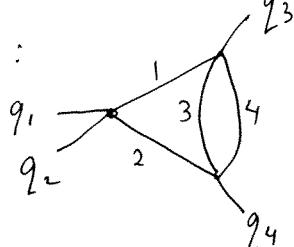
Defn :  $G$  a graph with external legs, incertly momenta  $q_i$ .

$$\phi_G(q) = \sum_{S=T_1 \cup T_2} \prod_{e \in S} \alpha_e \cdot (q^S)^2$$

$S$  = spanning 2-tree : subgraph  $S \subseteq G$  st  $S$  has 2 components  $T_1, T_2$  which are trees, &  $S$  meets every vertex of  $G$ .

$q^S = \pm q^{T_1} = \pm q^{T_2}$  total momentum entry  $T_1$  (or  $T_2$ ).

Ex :



$$\phi_G(q) = q_4^2 \alpha_2 \alpha_3 \alpha_4 + q_3^2 \alpha_1 \alpha_3 \alpha_4 + (q_1 + q_2)^2 (\alpha_1 \alpha_2 \alpha_4 + \alpha_1 \alpha_2 \alpha_3)$$

- $\deg \phi_G(q) = h_G + 1$

Key property :

$$\boxed{\phi_G(q) = \psi_\gamma \phi_{G/\gamma}(q) + R_\gamma(q)}$$

$R_\gamma$  of degree  $> h_\gamma$  in the  $\alpha_e$ -variables, e.g.

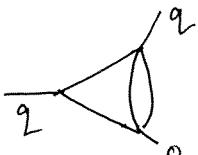
## (ii) 1-scale graphs

- Renormalize wrt a single chosen scale (tree definite quad. form in mere finite integrals)
- Angular dependence factor out as finite integrals (cf. Dirk's talk)
- Only need look at graphs with single scale. The Hopf algebra needs to be modified to put 1-scales on subgraphs  $\Leftrightarrow$  renormalization.

Defn: A 1-scale graph is a union of connected graphs, each component having a choice of exactly 2 distinguished vertices.



represents



Let  $G^\circ = G / \text{glue the two vertices}$  if  $G$  connected.

$$G^\circ = \begin{array}{c} 1 \\ \circ \\ 2(3)4 \end{array} \quad (\text{no longer 1-scale})$$

Claim:  $\phi_G(q) = q^2 \psi_{G^\circ}$ . trivial  $q$ -dependence.

Define:  $\phi_G = \psi_{G^\circ}$ .

$$\text{If } G = \bigcup_{i=1}^n G_i$$

$\psi_G := \prod \psi_{G_i}$ $\phi_G := \sum_{i=1}^n \phi_{G_i} \prod_{j \neq i} \psi_{G_j}$
---

Not the usual definition!!

Key property:

$$\frac{\phi_{G \cup G_i}}{\psi_{G \cup G_i}} = \sum \frac{\phi_{G_i}}{\psi_{G_i}}$$

additivity

(iii)  $\gamma$  and circular joins

Let  $\gamma, \Gamma$  be 1-scale graphs,  $\Gamma$  connected. A crucial role is played by

Defn : 
$$\mathcal{V}_{\gamma; \Gamma}(s) = s \psi_{\gamma} \phi_{\Gamma} + \phi_{\gamma} \psi_{\Gamma}$$
  $s = \text{scale.}$



$$\mathcal{V}_{\gamma; \Gamma}(s) = s \underbrace{(\alpha_3 + \alpha_4)}_{\psi_{\gamma}} \underbrace{\alpha_1 \alpha_2}_{\phi_{\Gamma}} + \underbrace{(\alpha_1 + \alpha_2)}_{\psi_{\Gamma}} \underbrace{\alpha_3 \alpha_4}_{\phi_{\gamma}}$$

When  $s=1$ ,  $\mathcal{V}_{\gamma; \Gamma}(1) = \phi_{\gamma \cup \Gamma}$ .

Rem : "Everything" reduces to ordinary graph polynomials when  $s=1$ .

Eq :  $\phi_{\gamma_1 \cup \dots \cup \gamma_n} = \psi_c(\gamma_1 \cup \dots \cup \gamma_n)$   $c = \text{circular join}$

$$\gamma_1 \circledast \gamma_2 \circledast \dots \circledast \gamma_n$$

$$c(\gamma_1, \dots, \gamma_n) = \text{join in a circle by gluing marked vertices in any way.}$$

2). Feynman rules for 1-scale graphs.

The counter-terms in BPHZ depend on two graphs  $\gamma, \Gamma$  (typically  $\Gamma = G/\gamma$ ). So we need to construct a differential form depending on two 1-scale graphs  $\gamma, \Gamma$ .

Defn :  $\gamma, \Gamma$  labelled 1-scale graphs,  $\Gamma$  connected & labels disjoint.

$$\omega_{\gamma \otimes \Gamma}(s) := \frac{s \phi_{\Gamma}}{\psi_{\gamma} \psi_{\Gamma}^2 \mathcal{V}_{\gamma; \Gamma}(s)} \mathcal{L}_{\gamma \cup \Gamma}$$

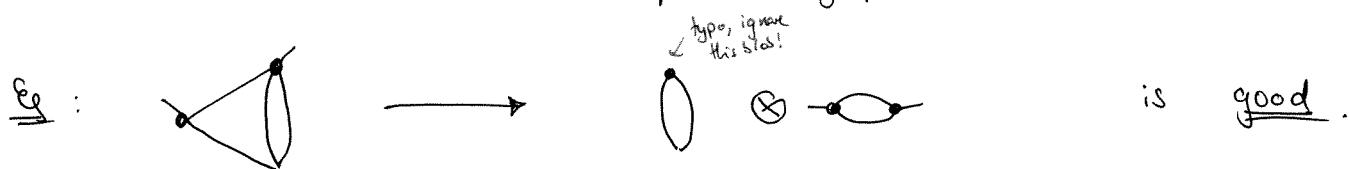
$$\deg \omega_{\gamma \otimes \tau}(s) = \text{sd}(\gamma) + \text{sd}(\tau).$$

If  $\Gamma$  is empty graph,

$$\omega_{\gamma \otimes \emptyset}(s) = 0 \quad \omega_{\emptyset \otimes \tau}(s) = \omega_\tau = \frac{\sqrt{2}}{4\tau^2}$$

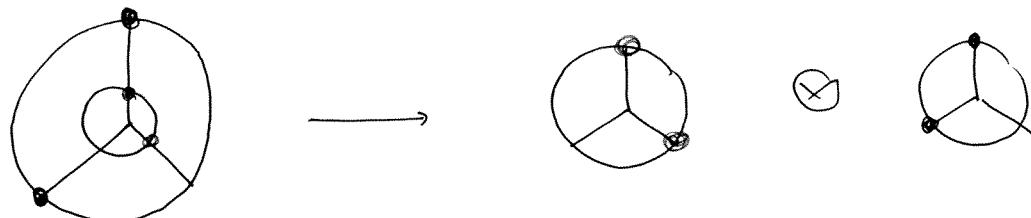
## ii). Melt algebras of 1-scale graphs.

For graphs with subdivisions, we want the 1-scale structures to transfer to the sub- and quotient-graphs in the coproduct.



In the first example, the inherited 1-scale structure on the cograph is a valid 1-scale (distinct vertices), in the second, the inherited structure is degenerate (2 marked vertices coalesce).

The 1-scale structure on the subgraph depends on a choice of momentum flow. To cut a long story short, we can always choose such 1-scale structures on subgraphs such that the "bad" coalescence of marked vertices in the coproduct never happens. For this talk, write the 1-scale structures on subgraphs with slots also, e.g.



Upshot: The set of such graphs (with labels on edges too) gives a Melt algebra  $H$ ,  $S:H \rightarrow H \otimes H$ , and furthermore it suffices to normalize these graphs only (cycle-dependencies are fine)

### 3). Renormalized amplitudes.

$G \in H$ , Hopf algebra of 1-scale graphs, call scale  $s$ .

$$\Delta : H \longrightarrow H \otimes H$$

Want to define renormalized amplitude

$$I_G^{\text{ren}} = \int \omega_G^{\text{ren}}(s)$$

#### (i) The preparation map.

Let  $H = \bigoplus_{n \geq 0} H_n$  be a graded Hopf algebra s.t.  $H_0 = \mathbb{Q}$ .  
 (connected). comute

$$\Delta' : H \longrightarrow H \otimes H \quad \text{reduced coproduct} \quad \Delta'(x) := \Delta(x) - x \otimes 1 - 1 \otimes x.$$

$$\Delta^{(n)} : H \longrightarrow H^{\otimes n} \quad \Delta^{(n)} = (\text{id} \otimes \Delta^{(n-1)}) \circ \Delta' = (\Delta^{(n-1)} \otimes \text{id}) \circ \Delta'$$

Sweedler :  $\Delta^{(n)}(x) = \sum x^{(1)} \otimes \dots \otimes x^{(n)}$  "decompose into  $n$  pieces,  
 $x^{(i)} \in H_{\geq i}$ ".

Refine : Preparation map

$$R : H \longrightarrow H \otimes_{\mathbb{Q}} H$$

$$R = 1 \otimes \text{id} + \sum_{n \geq 1} (-1)^n (\mu_n \otimes \text{id}) \Delta^{(n)}$$

Ex :  $\Delta x = 1 \otimes x + x \otimes 1 + x_1 \otimes x_2$

$$\Rightarrow R(x) = 1 \otimes x - x_1 \otimes x_2$$

(ii) Defn : Let  $G \in H$  & write  $RG = \sum a_i g_i \otimes G/g_i$

$$\boxed{\omega_G^{\text{ren}}(s) = \sum a_i \omega_{g_i \otimes G/g_i}(s)}$$

$$\text{Refine } f_T(s) = \int_{\Delta} \omega_T^{\text{rec}}(s)$$

Theorem: The integral converges.

$f_T(s)$  is  $s \frac{\partial}{\partial s}$  of the usual BPHZ-amplitude

Example 1:  $\Gamma$  subdivergence-free  $\Rightarrow \Delta\Gamma = 1 \otimes \Gamma + \Gamma \otimes 1$   
 $R\Gamma = 1 \otimes \Gamma$

$$\Rightarrow \omega_T^{\text{rec}}(s) = \omega_{1 \otimes \Gamma} = \frac{S_\Gamma}{4^2_\Gamma}$$

$$f_T(s) = \int \frac{S_\Gamma}{4^2_\Gamma}$$

gives back previous lecture.

Example 2:  $\Gamma = \begin{array}{c} 1 \\ \diagdown \quad \diagup \\ 2 \quad 3 \end{array} 4$   $\gamma = 3 \circ 4$

$$\Delta\Gamma = 1 \otimes \Gamma + \Gamma \otimes 1 + \gamma \otimes \Gamma/\gamma$$

$$R\Gamma = 1 \otimes \Gamma - \gamma \otimes \Gamma/\gamma$$

$$\omega_T^{\text{rec}}(s) = \frac{S_\Gamma}{4^2_\Gamma} - \omega_{\gamma \otimes \Gamma/\gamma}(s)$$

$$f_T(s) = \int_{\Delta} \left( \frac{1}{4^2_\Gamma} - \frac{\phi_{\Gamma/\gamma}}{4_\gamma^2 \psi_{\Gamma/\gamma}^2 \gamma_{\gamma; \Gamma/\gamma}(s)} \right) S_\Gamma$$

period.

We shall see later that all the information is contained in

$f_P(1)$ . Then the integrand can be entirely written in terms of graph polynomials  $\psi$  of  $\gamma, \Gamma/\gamma^0$ , and  $c(\gamma; \Gamma/\gamma)$ .

#### 4). Proof of finiteness (BPHZ theorem)

(i) Recall  $\mathcal{E}_I$  exceptional divisor above  $L_I \subseteq \mathbb{P}^{E_G-1}$  where  
 $L_I = \{(x_1 : \dots : x_{E_G}) \mid x_i = 0 \quad \forall i \in I\}$

We computed  $\text{Res}_{\mathcal{E}_I} \omega_G = \begin{cases} \omega_I \otimes \omega_{G/I} \\ 0 \end{cases}$

$$\text{sd}(I) = \dots$$

$$\text{sd}(I) < c$$

i.e.  $\omega_* : H \rightarrow \text{differential forms}$

$$\text{Res} = \bigoplus_I \text{Res}_{\mathcal{E}_I} \quad \text{total residue}$$

$$\boxed{\text{Res } \omega = (\omega \otimes \omega) \Delta}$$

Using the product structure for  $\psi$  and  $\phi$ , we similarly have

$$\boxed{\text{Res}_{\mathcal{E}_I} \omega_{\gamma \otimes G/\gamma}(s) = \begin{cases} \omega_{I_\gamma \cup I_{G/\gamma}} \otimes \omega_{\gamma/I_\gamma \otimes G/(G \cup I)}(s) \\ 0 \end{cases}}$$

where  $I_\gamma = I \cap \gamma$ ,  $I_{G/\gamma} = I_G / I_\gamma$ .

(ii) While  $\omega^{(2)} : H \otimes H \rightarrow \text{differential forms}$  for  $\omega_{\otimes}(s)$ .

Then

$$(1) \quad \omega^{(2)}(G \otimes I) = 0$$

$$(2) \quad \omega^{(2)}(I \otimes G) = \omega(G)$$

$$(3) \quad \text{Res } \omega^{(2)} = (\omega \otimes \omega^{(2)}) \circ \mu_{13} / \Delta \otimes \Delta$$

(If  $x_1 \otimes x_2 \otimes y_1 \otimes y_2 \in H^{\otimes 4}$

$$\text{then } \omega \otimes \omega^{(2)}(\mu_{13}(x_1 \otimes x_2 \otimes y_1 \otimes y_2)) = \omega(x_1 \otimes x_2) \otimes \omega^{(2)}_{x_2 \otimes y_2}.$$

The normalized Feynman rules are

$$\boxed{\omega^{\text{rel}} : H \rightarrow \text{diff. forms}}$$

$$\omega^{\text{rel}} = \omega^{(2)} \circ R.$$

(iii) Proof of finiteness

$$\begin{aligned}
 \text{Res } \omega^{\text{re}} &= \text{Res } \omega^{(2)} \circ R \stackrel{(3)}{=} (\omega \otimes \omega^{(2)}) \circ \mu_{13} (\Delta \otimes \Delta) \circ R \\
 &\stackrel{(1)}{=} (\omega \otimes \omega^{(2)}) \circ \underbrace{\mu_{13} (\Delta \otimes (\Delta - \text{id} \otimes 1))}_{1 \otimes R} \circ R \\
 &= (\omega \otimes \omega^{(2)}) \circ (1 \otimes R) \\
 &= 0 \quad \text{since } \omega(1) = 0
 \end{aligned}$$

$\Rightarrow \omega^{\text{re}}$  has no residues

$\Rightarrow$  integrand converges.  $\square$

We only need (2) to verify that leading term of  $\omega_0^{\text{re}}$  is  $\omega_G$ .

5). Graph by graph Callan-Symanzik equations.

Theorem:  $\Gamma$  a connected scale-scale graph in  $H$  (at most logarithmic sub-div.) , write  $\Delta' = \sum_\gamma \gamma \otimes T/\gamma$  reduced coproduct. Then we have the group equation



$$S \frac{\partial}{\partial S} f_\Gamma(s) = \sum_\gamma f_\gamma(1), f_{T/\gamma}(s)$$

$s > 0$ .

Example: For a single subdivergence  $\Gamma \subset T$

$$\Delta \Gamma = \Gamma \otimes 1 + 1 \otimes \Gamma + \gamma \otimes T/\gamma$$

CS eqn  $\Rightarrow$

$$f_\Gamma(s) = \underbrace{f_\Gamma(1)} + \underbrace{f_\gamma(1) f_{T/\gamma}(1)} \log s$$

Product of amplitudes of smaller graphs

A new number.

$$\Rightarrow \text{BPHZ amplitude} = f_\Gamma(1) \log s + f_\gamma(1) f_{T/\gamma}(1) \frac{\log^2 s}{2}$$

## Questions & comments

① We can define an MHS or motive for the renormalized integrand. What can we say about the position of  $[\omega_{\Gamma}^{\text{reg}}(s)]$  in the weight filtration?

Expect : graphs with subdivergences to have weight drop.

Q1: Is the "top-generic weight" part of a QFT renormalization-scheme independent?

Q2: What are the weights of angular integrals? (needed for Q1)

② ~~Amplitudes~~. I expect that  $\Gamma$  linearly-reducible  $\Rightarrow \int \omega_{\Gamma}^{\text{reg}}(s)$  can be computed parametrically.

③ What is a QFT? Renormalization requires very few inputs.

The key ingredients were:

① A Hopf algebra of graphs  $H$  (processes)

② For each  $G \in H$   $\xrightarrow{\quad}$  a pair of polynomials  $\psi_G, \phi_G(q)$   
 $\deg \phi_G = \deg \psi_G + 1$

③ Compatibility of ① & ② :

$$\begin{aligned}\psi_G &= \psi_g \psi_{G/g} + R_g \\ \phi_G(q) &= \phi_g \phi_{G/g}(q) + R_g\end{aligned}\quad \left\{ \begin{array}{l} \text{Product} \\ \text{structure} \end{array} \right.$$

①, ②, ③ are surprisingly rigid. Essentially  $H$  non-trivial forces  $\psi$  &  $\phi$  to be stably very close to graph polynomials considered above. Classify renormalizable QFT theories?